Chapter 15: Multiple Integrals

Section 1:

Definition 1:

R is a region in the plane.

$f(x,y)$ is a function of 2 variables which is defined on R (i.e. R is contained in Domain f), Then the double integral of f on R is defined as: $\iint_R f(x,y)\,dA(x,y) = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$

Theorem 1:

R is a region in the plane.

$f(x,y)$ is a function of 2 variables which is defined on R

$$\text{Then: } \left.\begin{array}{l} f(x,y) \text{ contiuous everywhere on } R \\ R \text{ is closed and bounded} \\ R \text{ has a "nice" boundary} \end{array}\right\} \Rightarrow \iint_R f(x,y)\,dA(x,y) \text{ exist.}$$

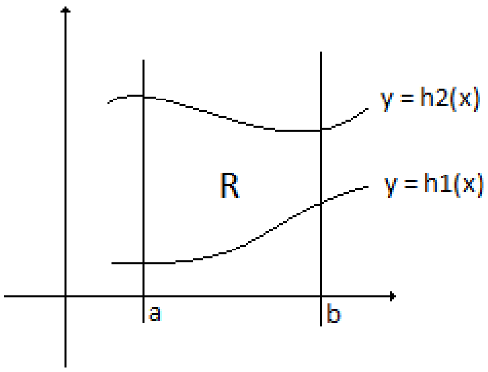
Theorem 2: Fubini's Theorem:

Suppose R is a closed and bounded region in the plane with nice boundary.

Suppose $f(x,y)$ is a function of 2 variables which is continuous everywhere in the R.

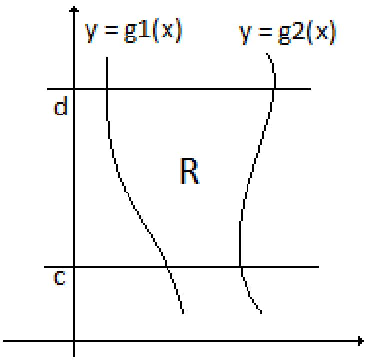
Then:

(i) If R looks like:



Then: $\iint_R f(x,y)\,dA(x,y) = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$ (Iterated Integral).

(ii) If R looks like:



Then: $\iint_R f(x,y)dA(x,y) = \int_c^d \int_{g_1(x)}^{g_2(x)} f(x,y)dxdy$

Section 2:

Definition 1:

Suppose R is a closed and bounded region in the plane with "nice" boundary.

The area of R is defined as: $\text{Area of } R = \iint_R dA$

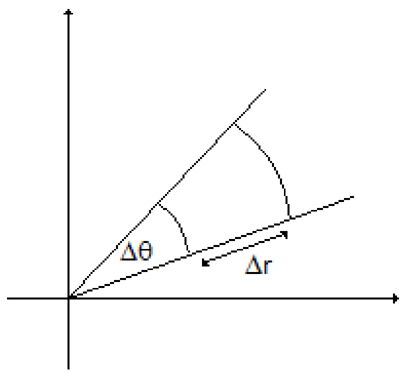
Definition 2:

R is a closed and bounded region in the plane with "nice" boundary.

Suppose $f(x,y)$ is a function of 2 variables which is continuous everywhere in R.

Then: The average of f on R is defined as: $\text{average value of } f \text{ on } R = \frac{1}{\text{area of } R} \iint_R f(x,y)dA(x,y)$

Section 3:



Notation : $\iint_R f(x,y)dA(x,y) = \iint_R f(x,y)dxdy$

Area of a polar rectangle of center (r,θ): $A = r\Delta r\Delta\theta$ **:**

Theorem: Changing to Polar Coordinates:

R is a closed and bounded region in xy-plane with "nice" boundary.

$f(x,y)$ is a function of 2 variables which is continuous everywhere in R.

Then: $\iint_R f(x,y)dxdy = \int_{R'} f(r\cos\theta, r\sin\theta)rdrd\theta$

Where R' is an appropriate region in the $r\theta$-plane.

Section 3:

Definition 1:

D is a region in the plane.

$f(x,y,z)$ is a function of 3 variables.

The triple integral of f on D is defined as:

$\iiint_D f(x,y,z)dV(x,y,z) = \lim_{\Delta V \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$ provided that the limit exists. Also,

$\Delta V = \max(\Delta V_1, \Delta V_2, \Delta V_3, \ldots\ldots, \Delta V_n)$

Theorem 1:

D is a region in the plane.

$f(x,y,z)$ is a function of 3 variables.

$$\text{Then: } \left.\begin{array}{l} f(x,y,z) \text{ contiuous everywhere on } R \\ D \text{ is closed and bounded} \\ D \text{ has a "nice" boundary} \end{array}\right\} \Rightarrow \iiint_D f(x,y,z)dV(x,y,z)$$

Section 5:

Definition 1:

Suppose D is a closed and bounded region in space with "nice" boundary.
Suppose S is a solid which occupies D.
Suppose $\delta(x, y, z)$ is the density of D.
Then:

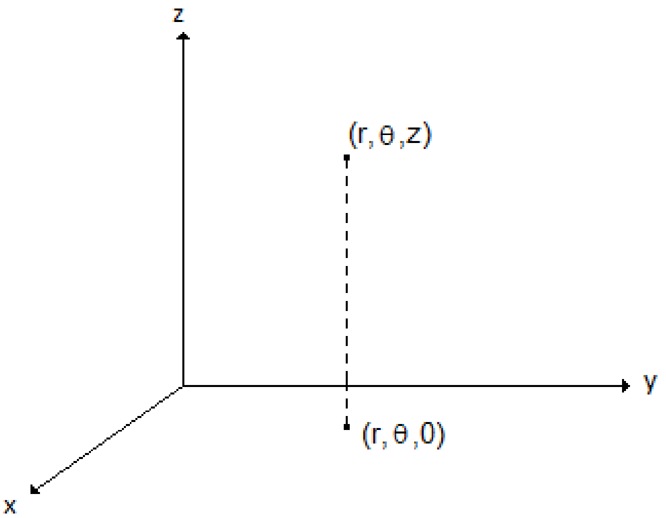
(i) The mass of S is $M = \iiint_D \delta(x, y, z)dV$

(ii) The moment of S about xy-plane is $\begin{cases} M_{xy} = \iiint_D z\ \delta(x, y, z)dV \\ M_{xz} = \iiint_D y\ \delta(x, y, z)dV \\ M_{yz} = \iiint_D x\ \delta(x, y, z)dV \end{cases}$

(iii)The coordinates of center of mass of the solid D are: $\begin{cases} \bar{x} = \dfrac{M_{yz}}{M} \\ \bar{y} = \dfrac{M_{xz}}{M} \\ \bar{z} = \dfrac{M_{xy}}{M} \end{cases}$

Section 6:

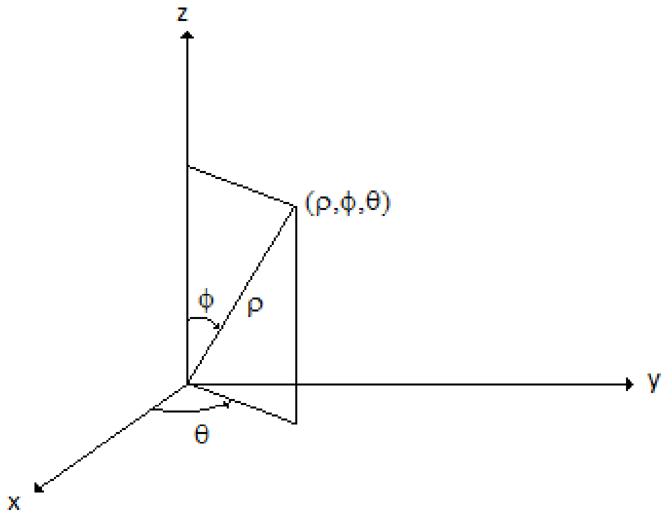
Definition 1: Cylindrical Coordinates:



Theorem 1: Changing to Polar Coordinates:

Suppose D is a region in space which is closed and bounded and has a "nice" boundary.
Suppose $f(x, y, z)$ is a function of 3 variables which is continuous everywhere id D
Then: $\iiint_D f(x, y, z)dxdydz = \iiint_{D'} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$ where D' is an appropriate region in the (r, θ, z) space.

Definition 2: Spherical Coordinates:



The characteristics of spherical coordinates.

$\rho \geq 0$

$0 < \phi < \pi$

$\rho = \sqrt{x^2 + y^2 + z^2}$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\sin\theta$

Theorem 2: Changing to Spherical Coordinates:
Suppose D is a region in space which is closed and bounded and has a "nice" boundary.
Suppose $f(x, y, z)$ is a function of 3 variables which is continuous everywhere id D
Then: $\iiint_D f(x, y, z)dxdydz = \iiint_{D'} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\phi d\theta$ where *D'* is an appropriate region in the (r, θ, z) space.

Section 7:

Theorem: Change of Variable formula:
R is a closed and bounded region in *xy*-plane with "nice" boundary.
$f(x, y)$ is a function of 2 variables which is continuous everywhere in *R*.
R' is a region in the *uv*-plane.

$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ A transformation from a *uv*-plane to the *xy*-plane that images *R'* one-to-one and onto

If the Jacobian $J(u,v) \neq 0$ everywhere in *R'*, then $\iiint_R f(x, y)dxdy = \iiint_{R'} f(g(u,v), h(u,v))|J(u,v)|dudv$

Where $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{dv} \\ \frac{\partial y}{du} & \frac{\partial y}{dv} \end{vmatrix}$